3-Dimensional Matching NP Completeness

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This serves as a simplified proof of the version presented in (Garey and Johnson 1990).

3DM Definition

Instance: set $M \subseteq W \times X \times Y$. W, X and Y are disjoint with q elements each.

Question: Does M contain a matching - a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

Theory. 3DM is NPC.

Proof.

Transform 3SAT to 3DM. Let $U = \{u_1, u_2, ..., u_n\}$ be the variables and $C = \{c_1, c_2, ..., c_m\}$ be the clauses. Construct disjoint sets W, X, Y such that $|W| = |X| = |Y|$ and set $M \subseteq W \times X \times Y$ such that M has a matching iff C is satisfiable.

Construct W,X,Y

Internal

$A = \{a_i[j] | 1 \leq i \leq n, 1 \leq j \leq m\}$

$B = \{b_i[j] | 1 \leq i \leq n, 1 \leq j \leq m\}$

$S_1 = \{s_1[j] | 1 \leq j \leq m\}$

$S_2 = \{s_2[j] | 1 \leq j \leq m\}$

$G_1 = \{g_1[j] | 1 \leq j \leq m(n-1)\}$

$G_2 = \{g_2[j] | 1 \leq j \leq m(n-1)\}$

External

$W = \{u_i[j], \overline{u_i}[j] | 1 \leq i \leq n, 1 \leq j \leq m\}$

$X = A \cup S_1 \cup G_1$

$Y = B \cup S_2 \cup G_2$

Note, each of W, X, Y contains $2mn$ elements, this being the number of matchings we must extract from M.

Construct M

Truth-setting and fan-out component

$T_i^t = \{(\overline{u_i}[j], a_i[j], b_i[j] | 1 \leq j \leq m\}$

$T_i^f = \{(u_i[j], a_i[(j+1)\%m], b_i[j]) | 1 \leq j \leq m\}$

$T_i = T_i^t \cup T_i^f$

Satisfaction testing component

$C_j = \{(u_i[j], s_1[j], s_2[j] | u_i \in c_j\} \cup \{(\overline{u_i}[j], s_1[j], s_2[j] | \overline{u_i} \in c_j\}$

Garbage collecting component

$G = \{(u_i[j], g_1[k], g_2[k]), (\overline{u_i}[j], g_1[k], g_2[k]) | 1 \leq k \leq m(n-1), 1 \leq i \leq n, 1 \leq j \leq m\}$

Combine the above into M

$M = (\bigcup_{i=1}^{n} T_i) \cup (\bigcup_{j=1}^{m} C_j) \cup G$

Reduction correctness

1. Reduction is constructed in polynomial time.
2. Each triple of M is an element of $W \times X \times Y$.
3. M contains a matching $\rightarrow$ C is satisfiable:
    1. The Matching $M' \subseteq M$ must include m triples from each T_i, either those corresponding to the variable u_i set to true or to false; u_i is set to true iff $M' \cap T_i = T_i^t$. Total number of matchings $= mn$.

2. Matching $M' \subseteq M$ must contain one triple from each C_j, since each triple of C_j shares variables $s_1[j]$ and $s_2[j]$. Hence we select one variable ($u_i[j]$ or $\overline{u_i}[j]$) per clause that doesn't occur in the triples of $M' \cap T_i$ selected in the previous step, which will be the case iff the select variable satisfies clause c_j. (Observe that the tuples of $M' \cap T_i$ comprise of the *negations* of those variables being set to true.) Total number of matchings $= m$.
3. We require *2mn* total matchings (one for each $u_i[j]$ and $\overline{u_i}[j]$), leaving us with $2mn - mn - m = m(n-1)$ to fulfill. The garbage collection component G serves this role. As the internal variables $g_1[k]$ and $g_2[k]$ are unique to this component, we select a unique $u_i[j]$ or $\overline{u_i}[j]$ from G not occuring in the remaining matchings of $M' - G$, provided the constraints in the previous two steps met.

4. C is satisfiable $\rightarrow M$ contains a matching:

   Let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for C, and let z_j be a literal set to true by t for clause $c_j \in C$. Construct $M' \subseteq M$ as follows:

   $M' = (\bigcup_{t(u_i)=T} T_i^t) \cup (\bigcup_{t(u_i)=F} T_i^f) \cup (\bigcup_{j=1}^{m} \{(z_j, s_1[j], s_2[j])\}) \cup G'$, where $G' \subseteq G$ is appropriately chosen to fill in the remaining number of matchings not selected from the truth-setting and satisfaction-testing components.

□

References

Garey, Michael R., and David S. Johnson. 1990. *Computers and Intractability; a Guide to the Theory of Np-Completeness.* New York, NY, USA: W. H. Freeman & Co.